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A covariant approach to braneworld holography

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Abstract

Exact holography for cosmological branes in an AdS-Schwarzschild bulk was first introduced in hep-th/0204218. We extend this notion to include all co-dimension one branes moving in non-trivial bulk spacetimes. We use a covariant approach, and show that the bulk Weyl tensor projected on to the brane can always be traded in for "holographic" energy-momentum on the brane. More precisely, a brane moving in a non-maximally symmetric bulk has exactly the same geometry as a brane moving in a maximally symmetric bulk, so long as we include the holographic fields on the brane. This correspondence is exact in that it works to all orders in the brane energy-momentum tensor.

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1 Introduction

Inspired by the entropy formula of black holes, the holographic principle asserts that there is a duality between gravity in n dimensions and a gauge theory in $n-1$ dimensions. The first concrete example of this was Maldacena's AdS/CFT correspondence, in which IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$ was found to be dual to $\mathcal{N}=4$ super Yang-Mills theory on the boundary [1, 2].

This remarkable idea can be studied in braneworld theories [3, 4]. In the single brane Randall-Sundrum model [3], we can think of gravity in the asymptotically anti de Sitter (AdS) bulk as being dual to a conformal field theory (CFT) on the brane/boundary [5]. The CFT has a UV cut-off and is coupled to gravity. In a very nice paper [6], Verlinde and Savonije examined cosmological branes moving in an n-dimensional AdS-Schwarzschild bulk. In the limit that the brane was close to the AdS boundary, they showed that the brane cosmology agreed with the standard cosmology in $(n-1)$ dimensions. Furthermore, the braneworld observer would see the bulk black hole *holographically* as dark radiation. Some time later, James Gregory and I noted that a holographic description held even when the brane was deep inside the bulk, far away from the boundary of AdS [7]. Briefly speaking, an observer living on an empty brane in an AdS Schwarzschild bulk experienced *exactly the same* evolution as an observer living on a non-empty brane in a maximally symmetric AdS bulk. The latter brane is non-empty in the sense that the fields of a dual gauge theory are sitting on the brane. The field theory is not conformal in general, although it approaches a CFT as the brane approaches the AdS boundary. Because of the remarkable exactness in the correspondence we found, we later dubbed this work *exact braneworld holography* [8].

In this paper, we will extend this notion of exact braneworld holography to include a much larger class of braneworlds. We will adopt a covariant approach to show that the geometry of any brane in any non-maximally symmetric bulk is the same as the geometry of a brane in a maximally symmetric bulk, provided we add some holographic matter to the brane. This holographic picture could be very useful in that we manage to entirely avoid the troublesome Weyl term projected on to the brane [9].

The rest of this paper is organised as follows: in the next section we briefly review exact holography for cosmological branes. In section 3, we show how exact holography is extended to arbitrary brane and bulk geometries using a covariant approach and the Brown and York (BY) stress-energy tensor [10]. In section 4, we discuss some properties of the holographic energy-momentum tensor, and suggest ways of calculating it explicitly. Section 5 contains some concluding remarks, and a discussion of the generalisation of this work to Lovelock gravities.

2 Exact holography for cosmological branes

We start by briefly reviewing precisely what we mean by exact holography for cosmological branes (for more details, see [7, 8]). Consider an $(n-1)$-dimensional brane

moving in a maximally symmetric n-dimensional AdS bulk. In global coordinates the bulk metric is given by

$$ds^2 = -V(a)dT^2 + \frac{da^2}{V(a)} + a^2 q_{ij} dx^i dx^j \tag{1}$$

where

$$V(a) = k^2 a^2 + 1 \tag{2}$$

and q_{ij} is the metric on a unit $(n-2)$-sphere. The brane is the following embedding in the bulk geometry

$$T = T(t), \qquad a = a(t), \qquad x^i = y^i \tag{3}$$

where

$$-V(a)\left[\dot{T}(t)\right]^2 + \frac{[\dot{a}(t)]^2}{V(a)} = -1 \tag{4}$$

This ensures that the brane metric is Friedmann-Robertson-Walker. Since the brane is cosmological, we assume that its energy-momentum is made up of tension, σ, and additional matter with energy density, ρ, and pressure, p. The Friedmann equation is [11, 12]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{2\Lambda_{n-1}}{(n-2)(n-3)} - \frac{1}{a^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho\left[1 + \frac{\rho}{2\sigma}\right]. \tag{5}$$

where Λ_{n-1} and G_{n-1} are the braneworld cosmological constant and Newton's constant respectively. Note that this takes the form of the $(n-1)$-dimensional standard cosmology when $\rho \ll \sigma$.

Now consider a brane with no additional matter, moving in an AdS black hole bulk. The bulk metric is now given by (1) with

$$V(a) = k^2 a^2 + 1 - \frac{\mu}{a^{n-3}}. \tag{6}$$

The black hole mass is proportional to μ. We can embed a cosmological brane in a similar way, and find that in this case the Friedmann equation is given by

$$H^2 = \frac{2\Lambda_{n-1}}{(n-2)(n-3)} - \frac{1}{a^2} + \frac{\mu}{a^{n-1}}. \tag{7}$$

In [7], we showed how we can calculate *exactly* the energy density of the black hole bulk, ρ_{holog}, measured by an observer on the brane – this can be done *without* assuming that the brane is near the AdS boundary. ρ_{holog} is given in terms of μ, so we can rewrite the Friedmann equation (7) to give

$$H^2 = \frac{2\Lambda_{n-1}}{(n-2)(n-3)} - \frac{1}{a^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho_{\text{holog}}\left[1 + \frac{\rho_{\text{holog}}}{2\sigma}\right]. \tag{8}$$

This takes exactly the same form as the Friedmann equation (5) for the brane moving in maximally symmetric AdS space with additional matter on the brane. We can

therefore think of ρ_{holog} as being the energy density of a field theory living on the brane. This field theory is dual to the AdS black hole bulk, although it is no longer conformal. We think of the dual field theory on the brane as being cut off in the ultra violet – this cutoff disappears as we go closer and closer to the AdS boundary, and we approach a conformal field theory. In this case, we are not assuming that the brane is near the boundary, so the cutoff can be significant.

3 Exact holography for all co-dimension one branes

We will now show the result reviewed in the previous section can be generalised to a much broader class of brane geometries. We make use of the covariant formalism of [9], and are able to generalise the notion of exact holography in a remarkably clear and simple way.

Consider an $(n-1)$-dimensional brane moving in an n-dimensional bulk. We will assume for simplicity that we have $\mathbb{Z}_2$ symmetry across the brane. This means that the brane splits the bulk into two identical domains. Each domain can be thought of as a manifold $\mathcal{M}$, with a boundary $\partial\mathcal{M}$ that coincides with the brane.

Now for some notation. The bulk metric is given by

$$ds^2 = g_{ab}dx^a dx^b \tag{9}$$

As in the previous section, we can think of the brane as an embedding in the bulk geometry

$$x^a = X^a(y^\mu). \tag{10}$$

We use this to determine the tangents to the brane

$$V^a_\mu = \frac{\partial X^a}{\partial y^\mu} \tag{11}$$

The induced metric on the brane is therefore given by

$$\gamma_{\mu\nu} = g_{ab}V^a_\mu V^b_\nu \tag{12}$$

We will also denote the normal to the brane by n^a. This enables us to define the brane extrinsic curvature

$$K_{\mu\nu} = \nabla_{(a}n_{b)}V^a_\mu V^b_\nu \tag{13}$$

We are now ready to define the action describing our braneworld scenario

$$S = 2S_g + S_m \tag{14}$$

where

$$S_g = M^{n-2}\left[\int_{\mathcal{M}} d^5x\sqrt{-g}\,(R(g) - 2\Lambda) + \int_{\partial\mathcal{M}} d^4y\sqrt{-\gamma}2K\right] \tag{15}$$

$$S_m = \int_{\text{brane}} d^4y\sqrt{-\gamma}L_m \tag{16}$$

Here M is the bulk Planck mass, and Λ is the bulk cosmological constant. There is no additional matter in the bulk, although there is an arbitrary matter distribution on the brane with Lagrangian L_m. Note that we have two copies of S_g in the action (14) because we have two copies of the bulk $\mathcal{M}$.

It is worth pointing out at this point that we have used the so-called "Trace-K" form for the gravitational part of the action S_g [13]. This is in keeping with our notion of the brane forming the boundary of the bulk spacetime. The Gibbons-Hawking term ensures that the correct bulk and brane equations of motion are obtained from varying the action with respect to the bulk and brane metrics respectively. This approach is in contrast to the more common approach used in the braneworld literature, where the brane is regarded as a delta-function source in the Einstein equations. The two approaches are entirely equivalent at the level of the equations of motion, as of course they should be. The distinction lies at the level of the action: the Gibbons-Hawking term is not required in the more common approach, whereas it is required in the "variational" approach we use here. We have chosen this "variational" approach because it enables us to see the generalisation of exact holography much more easily.

We now proceed with varying the action with respect to the bulk and brane metrics [14]. The bulk equations of motion are just the Einstein equations with a cosmological constant

$$\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = 0 \implies R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda g_{ab}. \tag{17}$$

whereas the brane equations of motion are the Israel equations

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma^{\mu\nu}} = 0 \implies 4M^{n-2}\left(K_{\mu\nu} - K\gamma_{\mu\nu}\right) = T^{(m)}_{\mu\nu} \tag{18}$$

where

$$T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_m}{\delta \gamma^{\mu\nu}} \tag{19}$$

Since we are mainly interested in the dynamics felt by an observer living on the brane, we will make use of the Gauss-Codazzi equations.

$$\mathcal{R}_{\mu\nu\alpha\beta}(\gamma) = R_{abcd}(g)V^a_\mu V^b_\nu V^c_\alpha V^d_\beta + K_{\mu\alpha}K_{\nu\beta} - K_{\mu\beta}K_{\nu\alpha} \tag{20}$$

$$D^\mu(K_{\mu\nu} - K\gamma_{\mu\nu}) = R_{ab}n^a V^b_\nu \tag{21}$$

where D_μ and $\mathcal{R}_{\mu\nu\alpha\beta}(\gamma)$ are the covariant derivative and Riemann tensor for the brane geometry. Given the Israel equations (18), and the fact that $R_{ab} \propto g_{ab}$, we can use the Codazzi equation (21) to show that energy on the brane is conserved

$$D^\mu T^{(m)}_{\mu\nu} = 0 \tag{22}$$

The Einstein equations (17) imply that the bulk Riemann tensor takes the following form

$$R_{abcd}(g) = C_{abcd} + \frac{2\Lambda}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{23}$$

where C_{abcd} is the bulk Weyl tensor. This vanishes when we have maximal symmetry. Inserting (23) into the Gauss equation (20), contracting, and making use of the Israel equations (18), we obtain the following formula for the brane Ricci tensor [9]

$$\mathcal{R}_{\mu\nu}(\gamma) = -E_{\mu\nu} + \frac{2\Lambda}{n-1}\gamma_{\mu\nu} - \left(\frac{1}{4M^{n-2}}\right)^2 \left[T^{(m)}_{\mu\alpha}T^{(m)\alpha}{}_{\nu} - \frac{T^{(m)}}{n-2}T^{(m)}_{\mu\nu}\right] \tag{24}$$

where $E_{\mu\nu} = C_{abcd}V^a_\mu n^b V^c_\nu n^d$ represents a non-local contribution coming from the bulk Weyl tensor. For the cosmological brane discussed in the previous section, it corresponds to the term proportional to μ in equation (7) [15]. It is this quantity that we would like to interpret holographically. Can we reinterpret it as some holographic fields living on the brane?

As in [7], the idea is that we calculate the energy-momentum of the bulk measured by an observer living on the brane. How might we go about doing this? It is well known that there is no local definition for the stress-energy-momentum (SEM) of the bulk gravitational field [10]. One needs to adopt a "quasi-local" definition on the boundary of a given region. Furthermore, if we wish to derive a global quantity, such as the total SEM in the bulk, one does so by considering the limit of the quasi-local SEM measured by observers on the boundary of the entire bulk. In our case, this boundary corresponds to the brane, so we immediately arrive at the bulk SEM measured by observers on the brane.

We will use Brown and York's definition for the quasi-local stress-energy tensor, $T^{\mathrm{BY}}_{\mu\nu}$ [10]. We believe this is a compelling definition since it enables us to associate the following conserved charge, with a Killing vector, ξ^μ on the boundary.

$$Q(\xi) = \int_S d^{n-2}\zeta\sqrt{\lambda}\; u^\mu T^{\mathrm{BY}}_{\mu\nu}\xi^\nu \tag{25}$$

where S is a spacelike surface lying in $\partial\mathcal{M}$, with normal u^μ, and induced metric

$$\lambda_{ij} = \gamma_{\mu\nu}\frac{\partial y^\mu}{\partial\zeta^i}\frac{\partial y^\nu}{\partial\zeta^j} \tag{26}$$

Making use of the Brown and York stress-energy tensor requires us to define a suitable background spacetime[1]. The background we choose satisfies the following two properties [16, 10, 7, 8]:

- the bulk, $\bar{\mathcal{M}}$, is maximally symmetric so that

$$\bar{R}_{abcd}(\bar{g}) = \frac{2\Lambda}{(n-1)(n-2)}(\bar{g}_{ac}\bar{g}_{bd} - \bar{g}_{ad}\bar{g}_{bc}) \tag{27}$$

- the boundary, or "cutoff" surface, $\partial\bar{\mathcal{M}}$, must have exactly the same geometry as the brane. In other words

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} \tag{28}$$

[1]From now on, we will label all background quantities with a "bar" , as will become obvious.

Some comments are in order here. Firstly, we believe that the choice of a maximally symmetric background is a natural one, and indeed the one that is most often used in calculating, say, the mass of a black hole spacetime. Furthermore, the Weyl term $\bar{C}_{abcd} = 0$, for this background. Recall that it is the Weyl term $E_{\mu\nu}$ in (24) that we are trying to understand holographically. It is appropriate that we should choose a background for which this term is absent.

Secondly, we have followed the presciption of [10, 16] in demanding that the background be cut-off at a surface whose induced metric is identical to the brane metric. As with the brane, we can think of the cutoff, $\partial\bar{\mathcal{M}}$, as a surface embedded in the background bulk. In principle it is not always possible to find an embedding with precisely the desired geometry. However, we have seen that it is always possible for the cosmological branes described in the previous section. We shall proceed under the assumption that a suitable cutoff surface can indeed be found.

Now that we have defined a background, we can define the *physical* action for the bulk [16]

$$S_{phys} = S_g - \bar{S}_g. \tag{29}$$

Here the background action is given by

$$\bar{S}_g = M^{n-2}\left[\int_{\bar{\mathcal{M}}} d^5x\sqrt{-\bar{g}}\left(\bar{R}(\bar{g}) - 2\Lambda\right) + \int_{\partial\bar{\mathcal{M}}} d^4y\sqrt{-\gamma}2\bar{K}\right] \tag{30}$$

We are now ready to calculate the BY stress-energy tensor of the bulk as measured by an observer on the brane

$$T^{\mathrm{BY}}_{\mu\nu} = -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{phys}}{\delta\gamma^{\mu\nu}} = -2M^{n-2}\left(K_{\mu\nu} - K\gamma_{\mu\nu}\right) + 2M^{n-2}\left(\bar{K}_{\mu\nu} - \bar{K}\gamma_{\mu\nu}\right) \tag{31}$$

We now associate this with the *holographic* energy-momentum tensor, $T^{(h)}_{\mu\nu} = T^{\mathrm{BY}}_{\mu\nu}$. Making use of the Israel equation (18), we see that

$$4M^{n-2}\left(\bar{K}_{\mu\nu} - \bar{K}\gamma_{\mu\nu}\right) = T^{(m)}_{\mu\nu} + 2T^{(h)}_{\mu\nu} \tag{32}$$

This equation corresponds to the Israel equation for the cutoff surface, $\partial\bar{\mathcal{M}}$ moving in the background bulk. Note that it is behaving like a brane containing the original matter, $T^{(m)}_{\mu\nu}$, plus some additional holographic matter, $T^{(h)}_{\mu\nu}$. There are two copies of the holographic matter because there were two copies of $\mathcal{M}$.

Because the background, $\bar{\mathcal{M}}$, is maximally symmetric, there is no bulk Weyl tensor, as we saw in equation (27). Therefore, the corresponding expression for the Ricci tensor on $\partial\bar{\mathcal{M}}$ will not contain a troublesome Weyl term like $E_{\mu\nu}$. The Ricci tensor on $\partial\bar{\mathcal{M}}$, or equivalently, the brane, can be expressed as

$$\mathcal{R}_{\mu\nu}(\gamma) = \frac{2\Lambda}{n-1}\gamma_{\mu\nu} - \left(\frac{1}{4M^{n-2}}\right)^2\left[T_{\mu\alpha}T^{\alpha}_{\nu} - \frac{T}{n-2}T_{\mu\nu}\right] \tag{33}$$

where

$$T_{\mu\nu} = T^{(m)}_{\mu\nu} + 2T^{(h)}_{\mu\nu} \tag{34}$$

In going from equation (24) to equation (33), we have traded the bulk weyl term $E_{\mu\nu}$ for some holographic matter, $T^{(h)}_{\mu\nu}$, on the brane. So it seems that we always have two equivalent pictures: we can either think of the brane as moving in a non maximally symmetric bulk, or we can think of the brane as moving in a maximally symmetric bulk, provided we include some additional holographic matter on the brane. The remarkable thing is that this correspondence is exact, in that it works to all orders in $T_{\mu\nu}$ in equation (33). In a delightfully simple way, we have seen how to extend exact braneworld holography to more general braneworld geometries.

4 On the holographic energy-momentum

A natural question to ask is: what do know about the holographic matter? Unfortunately, not a great deal. For an asymptotically AdS bulk in 5 dimensions, we might expect it to correspond to $\mathcal{N} = 4$ super Yang-Mills with the conformal invariance strongly broken. In general, however, all we can say is that it corresponds to some abstract quantum field theory. We *do* know that the holographic matter satisfies conservation of energy, $D^\mu T^{(h)}_{\mu\nu} = 0$. This follows from the Codazzi equation applied in the background. In addition, we can use the contracted Bianchi identity on the brane, $D^\mu \mathcal{G}_{\mu\nu}(\gamma) = 0$, to show that

$$T^\mu_\alpha D_{[\mu} S^\alpha_{\nu]} = 0 \tag{35}$$

where

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{n-2} T \gamma_{\mu\nu}, \qquad T_{\mu\nu} = T^{(m)}_{\mu\nu} + 2T^{(h)}_{\mu\nu} \tag{36}$$

This suggests that the holographic matter responds to changes in the original matter content. This is no surprise, as we would expect a change in $T^{(m)}_{\mu\nu}$ to cause a change in the bulk Weyl tensor. In any case, the formula (35) might offer an avenue towards learning more about the holographic matter.

In principle we can explicitly calculate $T^{(h)}_{\mu\nu}$ by inverting equation (33). This would give us the holographic energy-momentum in terms of $T^{(m)}_{\mu\nu}$, $\gamma_{\mu\nu}$, and $\mathcal{R}_{\mu\nu}$. If we wanted to relate this to the Weyl term in the original bulk we would simply make use of equation (24). Of course, such an inversion process is highly non-trivial. In a highly symmetric scenario such as those studied in [7], the inversion process is relatively simple. Otherwise, we could make use of a series expansion as we will now illustrate with an example.

Let us consider the n-dimensional version of a single brane Randall Sundrum model [3]. We have a negative cosmological constant in the bulk

$$\Lambda = -\frac{1}{2}(n-1)(n-2)k^2 \tag{37}$$

and a finely tuned brane tension

$$T^{(m)}_{\mu\nu} = -4M^{n-2}(n-2)k\gamma_{\mu\nu} \tag{38}$$

We shall now attempt to invert equation (33) by expanding the holographic energy-momentum tensor as a power series in k

$$T^{(h)}_{\mu\nu} = 4M^{n-2}k\sum_{N=1}^{\infty}\tau^{(N)}_{\mu\nu}k^{-2N} \tag{39}$$

Inserting (37), (38) and (39) into equation (33) yields the following

$$\begin{aligned}\mathcal{R}_{\mu\nu}(\gamma) &= (n-3)\left[\tau^{(1)}_{\mu\nu} - \frac{1}{n-3}\tau^{(1)}\gamma_{\mu\nu}\right] \\ &+ \sum_{N=1}^{\infty}k^{-2N}\Bigg\{(n-3)\left[\tau^{(N+1)}_{\mu\nu} - \frac{1}{n-3}\tau^{(N+1)}\gamma_{\mu\nu}\right] \\ &- \sum_{M=1}^{\infty}\left[\tau^{(M)\alpha}_{\mu}\tau^{(N+1-M)}_{\alpha\nu} - \frac{1}{n-2}\tau^{(M)}\tau^{(N+1-M)}_{\mu\nu}\right]\Bigg\}\end{aligned} \tag{40}$$

Equating coefficients of powers of k, we find that

$$\tau^{(1)}_{\mu\nu} = \frac{1}{n-3}G_{\mu\nu}(\gamma) \tag{41}$$

and for $N \geq 1$, we get the recurrence relation

$$\begin{aligned}\tau^{(N+1)}_{\mu\nu} &= \frac{1}{n-3}\sum_{M=1}^{\infty}\Bigg[\tau^{(M)\alpha}_{\mu}\tau^{(N+1-M)}_{\alpha\nu} - \frac{1}{2}\tau^{(M)\alpha}_{\beta}\tau^{(N+1-M)\beta}_{\alpha}\gamma_{\mu\nu} \\ &-\frac{1}{n-2}\tau^{(M)}\tau^{(N+1-M)}_{\mu\nu} + \frac{1}{2(n-2)}\tau^{(M)}\tau^{(N+1-M)}\gamma_{\mu\nu}\Bigg]\end{aligned} \tag{42}$$

Using (41), and the recurrence relation (42), we can calculate the holographic energy-momentum tensor to whatever order we desire. We present the result here to second order

$$\begin{aligned}T^{(h)}_{\mu\nu} &= \frac{2M^{n-2}k}{n-3}\Bigg\{G_{\mu\nu}(\gamma) + \frac{1}{(n-2)^2k^2}\Bigg[\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}_{\nu} - \frac{1}{2}\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}\gamma_{\mu\nu} \\ &-\frac{n-1}{2(n-2)}\mathcal{R}\mathcal{R}_{\mu\nu} + \frac{n+1}{8(n-2)}\mathcal{R}^2\gamma_{\mu\nu}\Bigg]\Bigg\} + \mathcal{O}(k^{-3})\end{aligned} \tag{43}$$

If the holographic matter corresponded to a conformal field theory, we would expect the trace of the energy-momentum to vanish. However, in this case, the CFT is broken because the brane does not lie on the boundary of AdS. Taking the trace of equation (43) gives

$$T^{(h)} = M^{n-2}k\Bigg\{-\mathcal{R} - \frac{1}{(n-2)^2k^2}\left[\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta} - \frac{n-1}{4(n-2)}\mathcal{R}^2\right]\Bigg\} + \mathcal{O}(k^{-3}) \tag{44}$$

Although the expression (43) enables $T^{(h)}_{\mu\nu}$ to be determined locally on the brane, we can use (24) to substitute $\mathcal{R}_{\mu\nu} = -E_{\mu\nu}$. $E_{\mu\nu}$ is really a non-local quantity determined by the bulk equations of motion. The aforementioned substitution will therefore give us a non-local expression for $T^{(h)}_{\mu\nu}$, as we might have expected. In addition, since $E_{\mu\nu}$ is traceless, we have $\mathcal{R} = 0$. This means that the order k^{-1} term in equation (44) corresponds to the trace anomaly for the (slightly broken) CFT [17].

For $n = 5$, the bulk equations of motion have been solved order by order to derive the solution for $E_{\mu\nu}$ [18, 19]. This solution is actually made up of a combination of both local and non-local pieces. In [18, 19], the explicitly non-local piece is taken to be the holographic energy momentum. In contrast, we claim that the holographic energy momentum should be given by the BY stress-energy tensor, for the reasons discussed in the previous section.

5 Discussion

In this paper we have shown how exact holography can be extended to a general class of braneworld geometries. On the one hand we can think of a brane moving in a non-maximally symmetric bulk, whereas on the other hand we can think of a brane moving in a maximally symmetric bulk, but with some additional holographic matter on the brane. The correspondence is exact. We can always trade a non-trivial bulk geometry for some holographic fields on the brane. In this way, we can always avoid the troublesome Weyl term in equation (24). This might turn out to be the most useful aspect of this holographic picture

An interesting consequence of the arguments used in this paper is that they can trivially be extended to other gravity theories, such as Lovelock gravity [20]. In each case, if we make use of a generalised Brown and York stress-energy tensor, a holographic description should hold for co-dimension one branes. In [21], we studied cosmological branes moving in a background of Gauss-Bonnet black holes. One of our conclusions was that there was no version of *exact* braneworld holography, although an approximate version did exist. I now believe this conclusion may have been wrong. This is because we made use of the Gauss-Bonnet Hamiltonian [22] to evaluate the energy density in the bulk according to an observer on the brane.

Let us discuss this a little further. Consider Gauss-Bonnet gravity described by an action S, including all the appropriate boundary terms [23]. Now suppose we wish to calculate the quasi-local gravitational energy measured on the boundary $\partial\mathcal{M}$ of some spacetime region, $\mathcal{M}$. We can either use the Hamiltonian, or a generalised BY stress-energy tensor. The latter is given by the variation of the action with respect to the boundary metric

$$T^{\mathrm{BY}}_{\mu\nu} = -\frac{2}{\sqrt{-\gamma}}\frac{\partial S}{\partial\gamma^{\mu\nu}} \tag{45}$$

Given a Killing vector, ζ^μ, on $\partial\mathcal{M}$, we can still find an associated the conserved charge given by equation (25). As with Einstein gravity, I believe this is a compelling reason to adopt the Brown and York approach. Motivated by (25), we follow [10],

and suggest the following formula for the energy associated with time t.

$$E = \int_{S_t} d^{n-2}\xi \sqrt{\lambda}\; u^\mu T^{\rm BY}_{\mu\nu} t^\nu \tag{46}$$

where S_t are surfaces of constant t in $\partial\mathcal{M}$, and $t^\mu \frac{\partial}{\partial y^\mu} = \frac{\partial}{\partial t}$. Note that if we split t^μ into its lapse function and shift vector

$$t^\mu = N u^\mu + N^\mu \tag{47}$$

it can be shown that

$$E = -\int_{S_t} d^{n-2}\xi N \frac{\partial S}{\partial N} + N^\mu \frac{\partial S}{\partial N^\mu} = \int_{S_t} d^{n-2}\xi N \frac{\partial H}{\partial N} + N^\mu \frac{\partial H}{\partial N^\mu} \tag{48}$$

where H is the Hamiltonian (see [10, 24] for details of the Einstein gravity case). Now, in Einstein gravity, the Hamiltonian evaluated on a solution is given by

$$H = \int_{S_t} d^{n-2}\xi N \frac{\partial H}{\partial N} + N^\mu \frac{\partial H}{\partial N^\mu} \tag{49}$$

In other words, H is linear in N and N^μ. This means that the BY approach, and the Hamiltonian approach agree on the value of the energy. However, equation (49) does *not* hold for Gauss-Bonnet gravity. This is because, for Gauss-Bonnet gravity, the surface terms in H depend on the extrinsic curvature of S_t in $\partial\mathcal{M}$, and as a result, are non-linear in N [22].

We ought to stress, however, that even in Gauss-Bonnet gravity, the BY approach and the Hamiltonian approach agree on the mass of black holes. This is due to the presence of a timelike Killing vector. When t^μ is Killing, we choose S_t so that $t^\mu = N u^\mu$, and the extrinsic curvature of S_t in $\partial\mathcal{M}$ vanishes. This eliminates the source of any disagreement between the two approaches, thereby explaining why they both give the same value for the black hole mass. In contrast, a dynamical brane is generically moving around, and there will be no timelike Killing vector. This means the (non-conserved) BY energy will differ from the (non-conserved) Hamiltonian energy.

Given the fact that even in Gauss-Bonnet gravity, we are still able to define a conserved charge from the BY stress-energy tensor and a Killing vector (time-like or space-like), we believe that the Brown and York approach is more reliable, although the disagreement certainly deserves further investigation.

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References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231–252, `hep-th/9711200`.

[2] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253–291, hep-th/9802150.

[3] L. Randall and R. Sundrum, "An alternative to compactification," Phys. Rev. Lett. **83** (1999) 4690–4693, hep-th/9906064.

[4] A. Karch and L. Randall, "Locally localized gravity," JHEP **05** (2001) 008, hep-th/0011156.

[5] S. S. Gubser, "AdS/CFT and gravity," Phys. Rev. **D63** (2001) 084017, hep-th/9912001.

[6] I. Savonije and E. P. Verlinde, "CFT and entropy on the brane," Phys. Lett. **B507** (2001) 305–311, hep-th/0102042.

[7] J. P. Gregory and A. Padilla, "Exact braneworld cosmology induced from bulk black holes," Class. Quant. Grav. **19** (2002) 4071–4083, hep-th/0204218.

[8] A. Padilla, "Brane world cosmology and holography," hep-th/0210217.

[9] T. Shiromizu, K.-i. Maeda, and M. Sasaki, "The Einstein equations on the 3-brane world," Phys. Rev. **D62** (2000) 024012, gr-qc/9910076.

[10] J. D. Brown and J. York, James W., "Quasilocal energy and conserved charges derived from the gravitational action," Phys. Rev. **D47** (1993) 1407–1419.

[11] P. Binetruy, C. Deffayet, and D. Langlois, "Non-conventional cosmology from a brane-universe," Nucl. Phys. **B565** (2000) 269–287, hep-th/9905012.

[12] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, "Brane cosmological evolution in a bulk with cosmological constant," Phys. Lett. **B477** (2000) 285–291, hep-th/9910219.

[13] G. W. Gibbons and S. W. Hawking, "Action integrals and partition functions in quantum gravity," Phys. Rev. **D15** (1977) 2752–2756.

[14] H. A. Chamblin and H. S. Reall, "Dynamic dilatonic domain walls," Nucl. Phys. **B562** (1999) 133–157, hep-th/9903225.

[15] S. Mukohyama, T. Shiromizu, and K.-i. Maeda, "Global structure of exact cosmological solutions in the brane world," Phys. Rev. **D62** (2000) 024028, hep-th/9912287.

[16] S. W. Hawking and G. T. Horowitz, "The Gravitational Hamiltonian, action, entropy and surface terms," Class. Quant. Grav. **13** (1996) 1487–1498, gr-qc/9501014.

[17] T. Shiromizu and D. Ida, "Anti-de Sitter no hair, AdS/CFT and the brane-world," Phys. Rev. **D64** (2001) 044015, hep-th/0102035.

[18] S. Kanno and J. Soda, "Brane world effective action at low energies and AdS/CFT," Phys. Rev. **D66** (2002) 043526, `hep-th/0205188`.

[19] S. de Haro, K. Skenderis, and S. N. Solodukhin, "Gravity in warped compactifications and the holographic stress tensor," Class. Quant. Grav. **18** (2001) 3171–3180, `hep-th/0011230`.

[20] D. Lovelock, "The Einstein tensor and its generalizations," J. Math. Phys. **12** (1971) 498–501.

[21] J. P. Gregory and A. Padilla, "Braneworld holography in Gauss-Bonnet gravity," Class. Quant. Grav. **20** (2003) 4221–4238, `hep-th/0304250`.

[22] A. Padilla, "Surface terms and the Gauss-Bonnet Hamiltonian," Class. Quant. Grav. **20** (2003) 3129–3150, `gr-qc/0303082`.

[23] R. C. Myers, "Higher derivative gravity, surface terms and string theory," Phys. Rev. **D36** (1987) 392.

[24] L. B. Szabados, "Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article," Living Rev. Relativity **7** (2004) 4. http://www.livingreviews.org/lrr-2004-4 (cited on March 7th, 2006).